On the breaking of water waves of finite amplitude on a sloping beach

By H. P. GREENSPAN
Pierce Hall, Harvard University

(*Received* 6 *January* 1958)

SUMMARY

In a recent paper Carrier & Greenspan (1958) showed that, within the framework of the non-linear shallow-water theory, there exist waves which do *not* break as they climb a sloping beach. The formation of a shock or bore is dependent on a variety of factors (wave shape, particle velocity, etc.) and, as yet, no general criteria for breaking have been found. In this paper, we consider waves which propagate shoreward into quiescent water; it is shown that any compressive wave (a wave of positive amplitude) which has a non-zero slope at the wave-front eventually breaks before reaching the coastline. In fact, an explicit relation is obtained between the initial conditions and the position where breaking occurs.

The conservation equations of mass and momentum of the non-linear shallow-water theory are

$$[v^*(\eta^*+h^*)]_{x^*} = -\eta_{t^*}, \tag{1}$$

and

$$v^*_{t^*}+v^*v^*_{x^*} = -g^*\eta^*_{t^*}, \tag{2}$$

where the symbols η^*, h^*, and x^* are defined in figure 1, v^* is velocity, t^* is time, and g^* is the gravitational acceleration. The asterisks denote dimensional quantities. Let the depth be given by

$$h^* = \alpha(l_0^*-x^*),$$

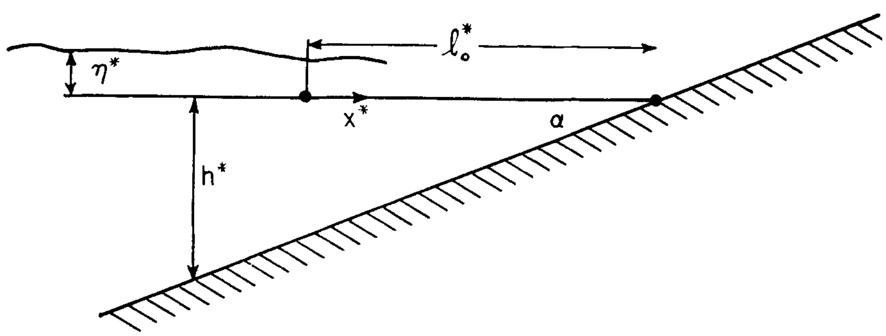

Figure 1. Fluid with a fixed boundary and a free surface.

and let $x = x^*/l_0^*$, $\eta = \eta^*/\alpha l_0^*$, $v = v^*/v_0^*$, $t = t^*/t_0^*$, where $t_0^* = (l_0^*/\alpha g^*)^{1/2}$ and $v_0^* = (\alpha g^* l_0^*)^{1/2}$. With the substitution of these dimensionless variables the basic equations become

$$[v(1-x+\eta)]_x = -\eta_t, \tag{3}$$

and

$$v_t + vv_x = -\eta_x. \tag{4}$$

These hyperbolic equations can be rewritten in a form in which the characteristics α, β play the role of independent variables and v, x, t and the wave velocity $c = (1-x+\eta)^{1/2}$ are unknown functions of α, β. The characteristic equations which arise are

$$x_\beta = (v+c)t_\beta, \tag{5}$$

$$(v+2c+t)_\beta = 0, \tag{6}$$

$$x_\alpha = (v-c)t_\alpha, \tag{7}$$

and

$$(v-2c+t)_\alpha = 0. \tag{8}$$

The reader is referred to Stoker (1948) for a complete derivation of these equations. Two families of characteristics are described. The quantity $v+2c+t$ is constant along any characteristic $\alpha =$ const. which propagates shoreward; the slope of this characteristic at any point (x,t) on it is $dx/dt = v+c$. The quantity $v-2c+t$ is constant along any characteristic $\beta =$ const. propagating seaward; the slope of the characteristic at any point on it is $dx/dt = v-c$.

Consider, then, waves which are propagating shoreward into quiescent water, and which at time $t=0$ are given by $\eta = 0$ for $0 \leqslant x \leqslant 1$, and $\eta = f(x)$ for $x \leqslant 0$, where $f(x)$ is a known function and $f(0) = 0$. The characteristic forming the wave-front can be determined explicitly. Since the wave is propagating into a region of rest, the particle velocity v is identically zero at any point on this characteristic. Therefore the slope of this curve is given by $dx/dt = c$. The quantity $v+2c+t$ is constant along this characteristic so that $v+2c+t = 2c+t = 2$, since $c=1$ at $x=0$ and $t=0$. This implies that the wave-front characteristic is the curve given by $dx/dt = 1-\frac{1}{2}t$, with $x=0$ at $t=0$; that is, the branch of the parabola $x = t-\frac{1}{4}t^2$ for which $t \leqslant 2$.

The wave velocity at the wave-front is

$$c = (1-x)^{1/2} = 1-\tfrac{1}{2}t. \tag{9}$$

At zero time the wave-front is located at the origin of the coordinate system fixed in the fluid. At a subsequent time t the wave-front has moved a distance $\int_0^t c\,dt = t-\frac{1}{4}t^2$ from the fixed coordinate system. If ξ measures distances from the moving wave-front (see figure 2), then

$$x = \xi + \int_0^t c\,dt = \xi + t - \tfrac{1}{4}t^2. \tag{10}$$

The origin of the new coordinate ξ is the position of the wave-front. If we now introduce (10) into the basic equations (3) and (4), we find that

$$\begin{aligned}[u(1-\xi-t+\tfrac{1}{4}t^2+\eta)]_\xi &= -\eta_t + (1-\tfrac{1}{2}t)\eta_\xi \\ &= u_\xi(1-\xi-t+\tfrac{1}{4}t^2+\eta) + u(-1+\eta_\xi),\end{aligned} \tag{11}$$

and

$$u_t - (1-\tfrac{1}{2}t)u_\xi + uu_\xi = -\eta_\xi. \tag{12}$$

In the moving coordinate system, $\eta_t = 0$, $\eta_{tt} = 0$, $u_t = 0$ and $u_{tt} = 0$ at $\xi = 0$.

If we set $\xi = 0$ in equations (11) and (12), we obtain the result

$$\eta_\xi = (1-\tfrac{1}{2}t)u_\xi \qquad \text{at } \xi = 0. \tag{13}$$

By differentiating (11) and (12) with respect to t, we find that

$$\eta_{\xi t} = u_{\xi t}(1-\tfrac{1}{2}t) - \tfrac{1}{2}u_\xi \qquad \text{at } \xi = 0. \tag{14}$$

Similarly, by differentiating with respect to ξ, we find that

$$u_{\xi t}(1-\tfrac{1}{2}t) - 2u_\xi + 3(1-\tfrac{1}{2})u_\xi^2 = -\eta_{\xi t} \qquad \text{at } \xi = 0. \tag{15}$$

By eliminating $\eta_{\xi t}$ from (15) by means of (14), there results an equation for $u_\xi(0,t)$ alone:

$$(1-\tfrac{1}{2}t)u_{\xi t} = \tfrac{5}{4}u_\xi - \tfrac{3}{2}(1-\tfrac{1}{2}t)u_\xi^2 \qquad \text{at } \xi = 0. \tag{16}$$

The corresponding equation for $\eta_\xi(0,t)$ is

$$(1-\tfrac{1}{2}t)\eta_{\xi t} = \tfrac{3}{4}\eta_\xi - \tfrac{1}{2}\eta_\xi^2. \tag{17}$$

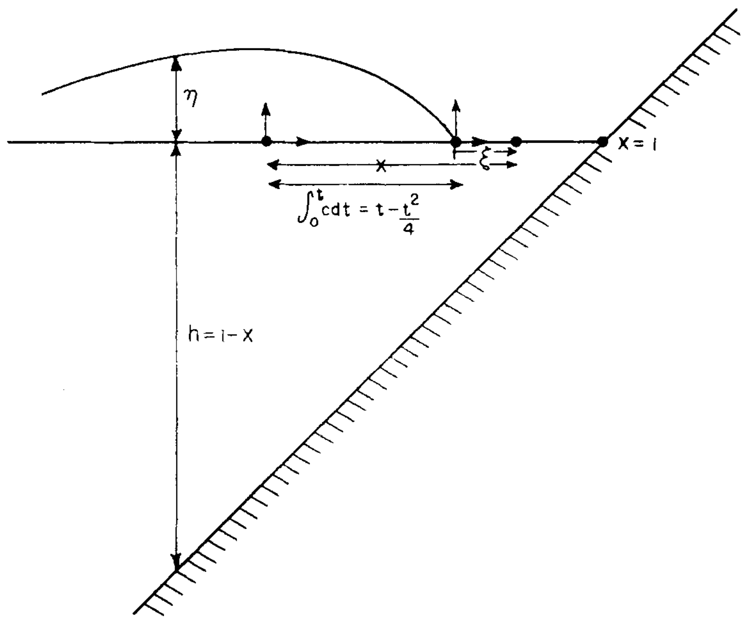

Figure 2. Location of the coordinate system fixed with respect to the wave-front.

Thus, if $\eta_\xi(0,0) < 0$, the wave is compressive at the wave-front, and $\eta_{\xi t}(0,0) < 0$ or the wave-front steepens. If the wave is initially compressive, with a non-zero slope at the wave-front, the wave-front steepens as it advances on the shoreline. On the other hand if $\eta_\xi(0,0) = 0$, then $\eta_\xi(0,t) = 0$ for all subsequent times. Such waves cannot begin to break or form a bore at the wave-front. Indeed, they may or may not break at some interior point. If we consider rarefaction waves for which $0 < \eta_\xi(0,0) < \frac{1}{2}$, we find that such waves actually steepen at the wave-front in contrast to rarefaction waves on a constant depth ocean, which always flatten out. A wave for which $\eta_\xi(0,0) = \frac{1}{2}$ continues to propagate shoreward with this slope at the wave-front. Waves for which $\eta_\xi(0,0) > \frac{1}{2}$ flatten out as they advance, and the slope at the wave-front approaches $\frac{1}{2}$.

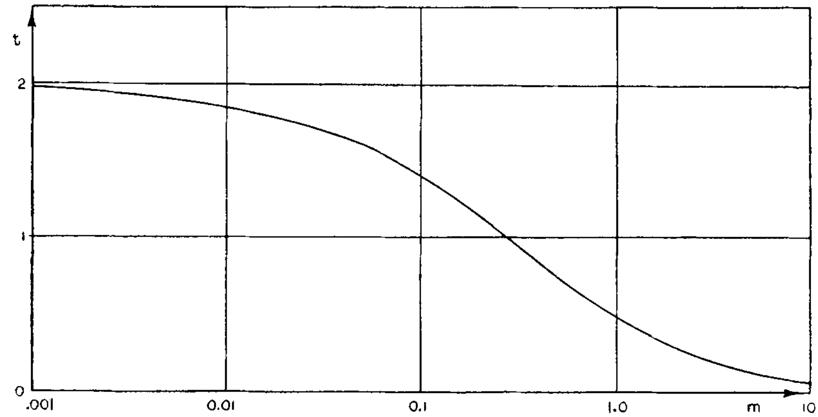

Figure 3. Breaking time as a function of initial slope.

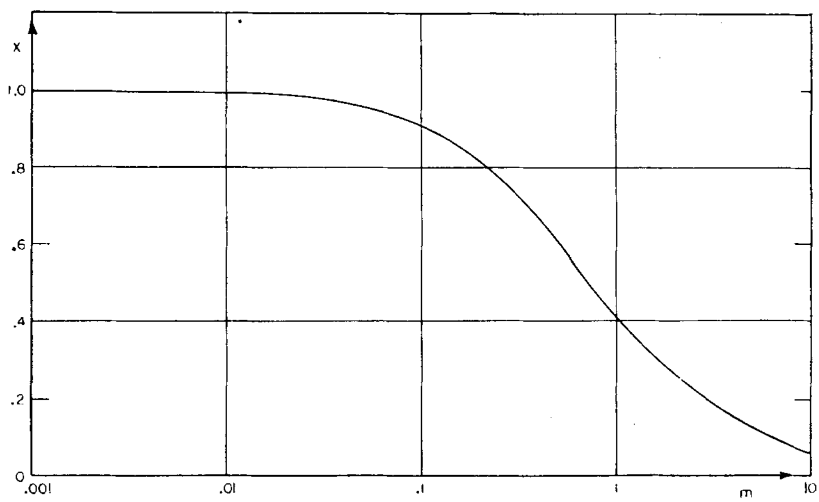

Figure 4. Breaking position as a function of initial slope.

Equations (16) and (17) are non-linear first-order differential equations for the functions $u_\xi(0,t)$ and $\eta_\xi(0,t)$ with the boundary conditions that $u_\xi(0,0)$ and $\eta_\xi(0,0)$ are specified. The solutions of these differential equations are

$$u_\xi(0,t) = 1/\{(2-t)[1-A^{1/2}(1-\tfrac{1}{2}t)^{3/2}]\}, \tag{18}$$

and

$$\eta_\xi(0,t) = 1/\{2[1-A^{1/2}(1-\tfrac{1}{2}t)^{3/2}]\}, \tag{19}$$

where

$$A = \left[\frac{u_\xi(0,0)-\frac{1}{2}}{u_\xi(0,0)}\right]^2 = \left[\frac{\eta_\xi(0,0)-\frac{1}{2}}{\eta_\xi(0,0)}\right]^2.$$

If $\eta_\xi(0,0) = -m$ where $m > 0$, then A is larger than one. A wave satisfying this condition breaks when the slope at the wave-front becomes infinite. From (19) it is seen that this occurs when

$$t = 2(1-A^{-1/3}) = 2\left\{1-\left[\frac{2m}{1+2m}\right]^{2/3}\right\} < 2,$$

or equivalently at $x = t-\frac{1}{4}t^2 < 1$.

Therefore waves which are compressive in the region adjacent to the wave-front and propagate shoreward with a discontinuity in slope eventually break before reaching the coastline. Values of t and x for which breaking occurs are plotted against m in figures 3 and 4.

Although general criteria for breaking are still lacking, the breaking point of a compressive wave with small radius of curvature near the wave-front can be accurately predicted.

This work was sponsored by the Office of Naval Research under Contract Nonr 1866(20).

References

Carrier, G. F. & Greenspan, H. P. 1958 Water waves of finite amplitude on a sloping beach, *J. Fluid Mech.* **4**, 97.

Stoker, J. J. 1948 The formation of breakers and bores, *Comm. Pure Appl. Math.* **1**, 9.